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PURSUIT OF A GROUP OF EVADERS IN THE PONTRYAGIN EXAMPLE†

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Sufficient conditions are derived for the capture of at least one evader for the Pontryagin example [1] with many participants and phase constraints imposed on the state of the evaders, with identical dynamic and inertial potentialities of the players, and with all the evaders using the same control.

The present paper is closely related to earlier investigations [2–12].

1. FORMULATION OF THE PROBLEM

In the space R^k ($k \geq 2$) we consider a differential game Γ of $n + m$ people: n pursuers $P_1, \dots, P_n$ and m evaders $E_1, \dots, E_m$ with the following laws of motion and initial conditions (at $t = 0$)

$$\begin{aligned} &x_i^{(l)} + a_1 x_i^{(l-1)} + \dots + a_l x_i = u_i, \quad \|u_i\| \leq 1 \\ &y_j^{(l)} + a_1 y_j^{(l-1)} + \dots + a_l y_j = \upsilon, \quad \|\upsilon\| \leq 1 \\ &x_i, y, u_i, \upsilon \in R^k, a_1, \dots, a_l \in R^1 \\ &x_i^{(\alpha)}(0) = x_{i\alpha}^0, \quad y_j^{(\alpha)}(0) = y_{j\alpha}^0, \quad \alpha = 0, \dots, l-1 \end{aligned} \tag{1.1}$$

where $x_{i0}^0 \neq y_{j0}^0$ for all i, j. Here and below, $i = 1, \dots, n, j = 1, \dots, m$. In addition it is assumed that the evaders E_j do not go beyond the limits of the convex set

$$D = \{y : y \in R^k, (p_s, y) \leq \mu_s, s = 1, \dots, r\}$$

where $p_1, \dots, p_r$ are unit vectors of R^k and $\mu_1, \dots, \mu_r$ are real numbers such that $\operatorname{Int} D \neq \varnothing$.

Definition 1. We will say that, in game Γ, capture occurs if an instant $T > 0$ and measureable functions

$$u_i(t) = u_i(t, x_{i\alpha}^0, y_\alpha^0, \upsilon(\cdot)), \quad \|u_i(t)\| \leq 1$$

exist such that, for any measurable function $\upsilon(t)$, $\|\upsilon(t)\| \leq 1, y_j(t) \in D, t \in [0, T]$, an instant of time $\tau \in [0, T]$ and numbers i, j exist such that $x_i(\tau) = y_j(\tau)$.

We will assume that $n \geq m$.

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2. AUXILIARY ASSERTIONS

Instead of system (1.1), we will examine the system

$$\begin{aligned} & z_{ij}^{(l)} + a_1 z_{ij}^{(l-1)} + \dots + a_l z_{ji} = u_i - \upsilon \\ & z_{ij}(0) = z_{ij0}^0 = x_{i0}^0 - y_{0}^0, \dots, z_{ij}^{(l-1)}(0) = z_{ijl-1}^0 = x_{il-1}^0 - y_{jl-1}^0 \end{aligned} \tag{2.1}$$

We will denote by $\varphi_p(t), p = 0, 1, \dots, l-1$ the solutions of the equation

$$w^{(l)} + a_1 w^{(l-1)} + \dots + a_l w = 0$$

with initial conditions

$$w(0) = 0, \dots, w^{(p-1)}(0) = 0, \quad w^{(p)}(0) = 1, \quad w^{(p+1)}(0) = 0, \dots, w^{(l-1)}(0) = 0$$

Proposition 1. All roots of the characteristic equation

$$\lambda^l + a_1 \lambda^{l-1} + \dots + a_l = 0 \tag{2.2}$$

have non-positive real parts.

Proposition 2. The function $\varphi_{l-1}(t)$ is non-negative for all $t \geq 0$.

Note that Proposition 2 is satisfied if Eq. (2.2) has only real roots. From Proposition 2 and a known result [10] it follows that Eq. (2.2) has at least one real root. We will denote by $\lambda_1, \dots, \lambda_s$ ($\lambda_1 <, \dots, < \lambda_s$) the real roots and by $\mu_1 \pm i\nu_1, \dots, \mu_q \pm i\nu_q$ ($\mu_1 \leq \mu_2 \leq \dots \leq \mu_q$) the complex roots of Eq. (2.2), by k_s the multiplicity of the root λ_s, and by m_α the multiplicity of the root $\mu_\alpha \pm i\nu_\alpha$. By virtue of Proposition 2, $\mu_q \leq \lambda_s$. Further, suppose

$$\begin{aligned} & (\eta_j(T,t), \zeta_i(T,t), \xi_{ij}(T,t)) = \varphi_0(T)(y_j(t), x_i(t), z_{ij}(t)) + \varphi_1(T)(\dot{y}_j(t), \dot{x}_i(t), \dot{z}_{ij}(t)) + \\ & + \dots + \varphi_{l-1}(T)(y_j^{(l-1)}(t), x_i^{(l-1)}(t), z_{ij}^{(l-1)}(t)) \end{aligned} \tag{2.3}$$

Then the functions (2.3) with $t = 0$ and the function $\varphi_{l-1}(t)$ can be represented in the form

$$\eta_j(T,0) = \Sigma_j^1(T), \quad \zeta_i(T,0) = \Sigma_i^2(T), \quad \xi_{ij}(T,0) = \Sigma_{ij}(T), \quad \varphi_{l-1}(t) = \Sigma^0(t)$$

Here

$$\begin{aligned} & \Sigma_m^n = \sum_{\beta=1}^{s} \exp(\lambda_\beta T) P_{m\beta}^n(T) + \sum_{\alpha=1}^{q} \exp(\mu_\alpha T)(Q_{m\alpha}^n(T)\cos\nu_\alpha T + R_{m\alpha}^n(T)\sin\nu_\alpha T) \\ & m = i, j; \quad n = 1, 2 \end{aligned} \tag{2.4}$$

The expression for $\Sigma_{ij}(T)$ differs from Eq. (2.4) in the absence of the superscript n and the replacement of m by ij, while the expression for $\Sigma^0(t)$ differs in the absence of the subscript m and the replacement of T by t, and for $\Sigma^0(t)$ $n = 0$.

We will assume that $\xi_{ij}(T, 0) \neq 0$ for all i, j and $t > 0$, for, if $\xi_{pq}(T, 0) = 0$ for certain p, q and T, then the pursuer P_p captures the evader E_q, assuming that $u_p(t) = \upsilon(t)$. We will also assume that $P_{ijs}(t) \neq 0$ for all i, j, as otherwise the pursuers initially aim to satisfy the given condition.

We will denote by γ_{ij} the degree of the polynomial P_{ijs}, and by γ the degree of the polynomial P_s^0. It can be assumed that $\gamma_{ij} = \gamma$ for all i, j, as otherwise the pursuers P_i initially aim to satisfy the given condition, selecting their controls $u_i(t)$ in a fairly small time interval so that the coefficients of t^γ of the polynomials P_{ijs} are non-zero.

Proposition 3. The inequality $m_\alpha < k_s$ holds for all $\alpha \in I = \{\alpha | \mu_\alpha = \lambda_s\}$.

We will put

$$X_i^0 = \lim\frac{P_{is}^2(t)}{t^\gamma},\quad Y_j^0 = \lim\frac{P_{js}^1(t)}{t^\gamma},\quad Z_{ij}^0 = \lim\frac{P_{ijs}(t)}{t^\gamma}\quad \text{as}\quad t\to\infty$$

$$C_{\alpha\beta}(T,t) = \eta_\alpha(T,t)-\eta_\beta(T,t) = C_{\alpha\beta}(T+t,0)$$

$$M_q(T,t,B_q(T+T^0)) =$$

$$= \exp(-\lambda_s(T^0+T))\int_0^t \varphi_{l-1}(T^0+t-\tau)\lambda(B_q(T^0+T),\upsilon(\tau))d\tau$$

We will define the function λ: $\text{comp}(R^k)\times V\to R$

$$\lambda(A,\upsilon) = \sup\{\lambda|\lambda>0, -\lambda A\cap(V-\upsilon)\neq\varnothing\}$$

Here $\text{comp}(R^k)$ is the space of convex compact subsets R^k with Hausdorff metrics, and V is a sphere of unit radius.

Lemma 1. Suppose Proposition 1–3 are satisfied, $D = R^k$, B_i: $[0,\infty)\to R^k$, $\inf_\upsilon\max_i\lambda\,(B_i(T^0+t),\upsilon)\geq\delta>0$ for all $t>0$. Then an instant $T>0$ exists such that, for any permissible function υ, a number q can be found such that

$$1-M_q(T,T,B_q(T+T^0))\leq 0$$

3. THE SUFFICIENT CONDITIONS FOR CAPTURE

We will assume that the initial conditions are such that:

(a) if $n>k$, then, for any set of subscripts $I\subset\{1,2,\ldots,n\}$, $|I|\geq k+1$, the condition $\text{Intco}\{X_i^0, i\in I\}\neq\varnothing$ holds;

(b) any k vectors from the set $\{X_i^0-Y_j^0, Y_l^0-Y_r^0, l\neq r\}$ are linearly independent.

Theorem 1. Suppose Propositions 1–3 are satisfied, $D = R^k$, $n\geq k+1$ and

$$0\in\text{Intco}\{Z_{ij}^0\}\tag{3.1}$$

Then, capture occurs in the game Γ.

Proof. From the condition of the theorem it follows that $n+m\geq k+2$. On the strength of a known result [11, Lemma 3] $I\subset\{1,\ldots,n\}$ and $J\subset\{1,\ldots,m\}$ exist such that $\{Z_{ij}^0, i\in I, j\in J\}$ form a positive basis and $|I|+|J| = k+2$. We will assume that

$$I = \{1,\ldots,q\},\quad J = \{1,\ldots,l\}$$

If $|J| = 1$, then capture follows from a known result [10]. We will assume that $|J|\geq 2$. From a known result [10, Lemma 2.4, p. 155] it follows that an instant $\hat{T}$ exists such that

$$\{\xi_{ij}(T^0+t,0), i\in I, j\in J\}\tag{3.2}$$

form a positive basis for any $T^0\geq\hat{T}$, $t\geq 0$. We will fix one of the given instants T^0. Since $\xi_{i\alpha}(T^0,t) = \xi_{i\alpha_0}(T^0,t)+C_{\alpha_0,\alpha}(T^0,t)$ for all $i\in I$, $\alpha\neq\alpha_0$, $\alpha\in J$, then

$$\{\xi_{i\alpha_0}(T^0+t,0), i\in I, C_{\alpha_0\alpha}(T^0+t,0)\ \alpha\neq\alpha_0, \alpha\in J\}$$

form a positive basis. Let $\alpha_0 = 1$. Then

$$\{\xi_{i1}(T^0+t,0), i\in I, C_{1\alpha}(T^0+t,0), \alpha\neq 1, \alpha\in J\}$$

form a positive basis, and here the number of vectors of the given set is equal to $k = 1$.

Since $n \geq k + 1$, subscripts $q + \alpha - 1 \in \{q + 1, \dots, n\}$ exist with $\alpha \in J$, $\alpha \neq 1$. From a known result in [10] it follows that a $\mu > 0$ exists such that the vectors

$$\{\xi_{i1}(T^0 + t, 0), i \in I, \xi_{q+\alpha-1 1}(T^0 + t) + \mu C_{1\alpha}(T^0 + t, 0), \alpha \in J, \alpha \neq 1\}$$

form a positive basis. Suppose

$$\Omega(t) = \{\upsilon_t(\cdot); \|\upsilon(\tau)\| \leq 1, \tau \in [0, t]\}$$

$$T(z_0) = \min\{t : t \geq 0, \inf_{\upsilon_t(\cdot) \in \Omega(t)} \max_{i,\alpha}(1 - h_i(t), 1 - h_{q+\alpha-1}(t)) \geq 1\}$$

where

$$h_\kappa(t) = 1 - M_\kappa(T, t, B_\kappa(T + T^0));$$

$$B_\kappa(T + T^0) = \exp(-\lambda_s(T^0 + T))(\xi_{\kappa 1}(T^0 + T, 0) + \mu_0 C_{1\alpha}(T^0 + T, 0));$$

$$\kappa = i \in I; \quad q + \alpha - 1; \quad \alpha \in J, \quad \alpha \neq 1$$

$$\mu_0 = \begin{cases} 0, & \kappa = i \in I \\ \mu, & \kappa = q + \alpha - 1; \quad \alpha \in J, \quad \alpha \neq 1 \end{cases}$$

From Lemma 1, $T(z_0) < \infty$.
We specify the controls of the pursuers P_i, assuming that ($T = T(z_0)$, $t \in [0, T]$)

$$u_\kappa(t) = \upsilon(t) - \lambda(B_\kappa(T^0 + T), \upsilon(t))B_\kappa(T^0 + T)$$

Let t_1 be the smallest positive root of the function h of the form $h(t) = \min_i h_i(t)$. We will assume that $u_i(t) = \upsilon(t)$, $t \in [t_1, T]$. Then

$$\xi_{\kappa 1}(T^0, t) + \mu_0 C_{1\alpha}(T^0, t) = \xi_{\kappa 1}(T^0 + t, 0) + \mu_0 C_{1\alpha}(T^0 + t, 0) +$$
$$+ (\xi_{\kappa 1}(T^0 + T, 0) + \mu_0 C_{1\alpha}(T^0 + T, 0))(h_\kappa(t) - 1)$$

By Lemma 1, for any function $\upsilon(\cdot)$, $\upsilon(\cdot) \in \Omega(T)$ a number r exists such that $h_r(T) = 0$.
If $r \in I$, then $\xi_{r1}(T^0, T) = 0$, and consequently, in game Γ, capture occurs at the instant of time $T^0 + T$, if we assume that $u_r(t) = \upsilon(T)$, $t \in [T, T^0 + T]$.
If $h_{q+\alpha_0-1 1}(T) = 0$ for certain $\alpha_0 \in J$, $\alpha_0 \neq 1$, then

$$\xi_{q+\alpha_0-1 1}(T^0, T) = -\mu C_{1,\alpha_0}(T^0, T) = -\mu C_{1\alpha_0}(T^0 + T, 0)$$

$$\xi_{i1}(T^0, T) = \xi_{i1}(T^0 + T, 0)h_i(t)$$

for all $i \in I$, and consequently

$$\{\xi_{ij}(T^0, T), i \in I, j \in J\}$$

form a positive basis. This means that

$$\zeta_i(T^0, T) - \eta_{\alpha_0}(T^0, T) + \eta_{\alpha_0}(T^0, T) - \eta_1(T^0, T), \quad \zeta_i(T^0, T) - \eta_j(T^0, T)\}$$

comprise a positive basis. Hence, it follows that

$$\{\xi_{ij}(T^0, T), i \in I, j \in J, j \neq 1, -C_{1\alpha_0}(T^0, T)\}$$

comprise a positive basis. Replacing $-C_{1\alpha_0}(T^0, T)$ by $\xi_{q+\alpha_0-1 1}(T^0, T)$, we obtain that, for any $T^0 > \hat{T}$

$$\{\xi_{ij}(T^0, T), i \in I \cup \{q+\alpha_0-1\}, j \in J, j \neq 1\}$$

comprise a positive basis. Consequently, the vectors

$$\{\xi_{ij}(T^0+t, T), i \in I \cup \{q+\alpha_0-1\}, j \in J, j \neq 1\} \tag{3.3}$$

form a positive basis for any $t \geq 0$.

Condition (3.3) is similar to condition (3.2), but in this case the number of evaders in condition (3.3) has been reduced by one. Taking the instant $T + T^0$ as the initial instant of time, and repeating the reasoning until the number of evaders becomes equal to one, we obtain that

$$\xi_{i1}(T^0+t, T), i \in I$$

form a positive basis for any $t \geq 0$, where $|I| = k + 1$. Hence, by virtue of the known result in [10], capture occurs in game Γ.

Theorem 2. Suppose Propositions 1–3 are satisfied, $n \geq k$, and

$$0 \in \operatorname{Intco}\{Z_{ij}^0, p_1, \ldots, p_r\}, \quad r \geq 1 \tag{3.4}$$

Then, capture occurs in game Γ.

4. EXAMPLES

Example 1. Systems (2.1) and (2.2) have the form

$$z_{ij}^{(4)} + \ddot{z}_{ij} = u_i - v, \quad \|u_i\| \leq 1, \quad \|v\| \leq 1$$

$$z_{ij}(0) = \overset{0}{z}_{ij}, \quad \dot{z}_{ij}(0) = \overset{1}{z}_{ij}, \quad \ddot{z}_{ij}(0) = \overset{2}{z}_{ij}, \quad z_{ij}^{(3)}(0) = \overset{3}{z}_{ij}$$

Then

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \varphi_2(t) = 1-\cos t, \quad \varphi_3(t) = t-\sin t$$

Therefore

$$\xi_{ij}(t, 0) = \varphi_0(t)\overset{0}{z}_{ij} + \varphi_1(t)\overset{1}{z}_{ij} + \varphi_2(t)\overset{2}{z}_{ij} + \varphi_3(t)\overset{3}{z}_{ij} =$$
$$= t(\overset{1}{z}_{ij} + \overset{3}{z}_{ij}) + (\overset{0}{z}_{ij} + \overset{2}{z}_{ij}) - \overset{2}{z}_{ij}\cos t + \overset{2}{z}_{ij}\sin t$$

We assume that $Z_{ij}^0 = Z_{ij}^1 + Z_{ij}^3$ and $Z_{ij}^0 \neq 0$.

Assertion. Suppose $n \geq k$ and condition (3.4) is satisfied. Then, capture occurs in game Γ.

Example 2. Systems (2.1) and (2.2) have the form

$$z_{ij}^{(l)} = u_i - v, \quad \|u_i\| \leq 1, \quad \|v\| \leq 1$$

$$z_{ij}^s(0) = \overset{s}{z}_{ij}, \quad s = 0, \ldots, l-1$$

Then

$$\varphi_s(t) = \frac{t^s}{s!}, \quad s = 0, \ldots, l-1$$

Therefore

$$\xi_{ij}(t, 0) = \sum_{s=0}^{l-1} \varphi_s(t) z_{ij}^s = \sum_{s=0}^{l-1} z_{ij}^s \frac{t^s}{s!}$$

We assume that $Z_{ij}^0 = Z_{ij}^{l-1}$ and $Z_{ij}^0 \neq 0$.

Assertion. Suppose $n \geq k$ and condition (3.4) is satisfied. Then capture occurs in game Γ.

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